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POLYNOMIAL APPROXIMATION IN THE MEAN WITH RESPECT TO HARMONIC MEASURE ON CRESCENTS

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ABSTRACT. For $1 \leq s < \infty$ and "nice" crescents G, this paper gives a necessary condition (Theorem 2.6) and a sufficient condition (Theorem 2.5) for density of the polynomials in the generalized Hardy space $H^s(G)$. These conditions are easily tested and almost equivalent.

The problem of polynomial approximation in the mean with respect to area measure on crescents has been extensively researched by such authors as J. Brennan [**1**] and S. Mergelyan [**6**]. This author explores the same problem except that area measure is replaced by harmonic measure. Hence, the question here may be rephrased: given $1 \leq s < \infty$, for which crescents G are the polynomials dense in the generalized Hardy space $H^s(G)$ (the collection of all analytic functions f on G for which $|f|^s$ has a harmonic majorant on G. For z_0 in G, $\|f\|_{z_0} = u_f(z_0)^{1/s}$, where u_f is the least harmonic majorant of $|f|^s$ on G)?

1. Preliminaries. Suppose G is a bounded Dirichlet region in the complex plane (that is, G is a bounded, open, connected set on which the Dirichlet problem is solvable). If $z \in G$, then define

$$\rho_z : C_{\mathbf{R}}(\partial G) \to \mathbf{R} \quad \text{by} \quad \rho_z(f) = \hat{f}(z),$$

where $\hat{f}$ is the solution to the Dirichlet problem on G with boundary values f. By the maximum principle, ρ_z is a bounded, positive linear functional on $C_{\mathbf{R}}(\partial G)$. Since ∂G is compact, the Riesz representation theorem gives a unique Borel measure $\omega \equiv \omega(\cdot, G, z)$ (indeed a probability measure by the maximum principle and the fact that $\rho_z(1) = 1$) with support$(\omega) \subseteq \partial G$ such that $\rho_z(f) = \int f\, d\omega$ for all f in $C_{\mathbf{R}}(\partial G)$. This measure is called *harmonic measure* for G at z. To familiarize oneself with some of the properties of harmonic measure one may consult [**4**]. Throughout this paper, "m" denotes Lebesgue measure on $\mathbf{R}$.

1.1 DEFINITION. A *crescent* is a region (in the complex plane $\mathbf{C}$) bounded by two Jordan curves which intersect in a single point such that one of the Jordan curves is internal to the other.

If G is a crescent, then denote the *outer boundary* of G by $\partial_\infty G \equiv \partial \overline{G}^\wedge$ ($\overline{G}^\wedge$ being the polynomially convex hull of the closure of G) and the *inner boundary*

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of G by $\overline{\partial_0 G}$, where $\partial_0 G \equiv \{\partial G\}\backslash\{\partial_\infty G\}$. The *multiple boundary point* of G is defined to be $\{\partial_\infty G\} \cap \{\overline{\partial_0 G}\}$. The reader should note that crescents are simply connected Dirichlet regions.

The following two propositions are primarily due to the relationship between conformal maps and harmonic measure. The proof of each is left as an exercise.

1.2 PROPOSITION. *Suppose that G is a crescent and ω is harmonic measure for G at z_0 in G. Let E be a subregion of G such that ∂E is a Jordan curve and $\Gamma \equiv \{\partial E\} \cap \{\partial G\}$ is a Jordan arc with endpoints α and β. Let ν be harmonic measure for E at ξ_0 in E. If K is a compact subset of $\Gamma\backslash\{\alpha,\beta\}$, then there exists a constant $r \equiv r(K) > 0$ such that $(1/r)\cdot\nu|_K \leq \omega|_K \leq r\cdot\nu|_K$.*

1.3 PROPOSITION. *Suppose that G is a crescent with multiple boundary point λ, and ω is harmonic measure for G at z_0 in G. If ν is harmonic measure for $\mathrm{int}(\overline{G}\,\hat{}\,)$ at z_0, then, for any compact subset K of $\{\partial_\infty G\}\backslash\{\lambda\}$, there exists a constant $r \equiv r(K) > 0$ such that $\omega|_K \leq \nu|_K \leq r\cdot\omega|_K$. Consequently, $\int_K \log(d\omega/d\nu)\,d\nu > -\infty$.*

If K is a compact subset of $\mathbf{C}$, σ is a regular Borel measure with support in K and $1 \leq s < \infty$, then $P^s(\sigma)$ [resp. $R^s(K,\sigma)$] denotes the closure of the polynomials [resp. the rational functions with the poles off K; Rat (K)] in $L^s(\sigma)$. If G is a crescent, ω is harmonic measure for G at z_0 in G and $1 \leq s < \infty$, then one can show that $R^s(\overline{G},\omega)$ is isometrically isomorphic to $H^s(G)$ (the collection of all analytic functions f on G such that $|f|^s$ has a harmonic majorant on $G \cdots \|f\|_{z_0} = u_f(z_0)^{1/s}$; u_f is the least harmonic majorant of $|f|^s$ on G).

2. For which crescents G is $P^s(\omega(\cdot, G, z_0)) = H^s(G)$? The primary focus of this paper is the following question, which is a paraphrase of the above title.

2.1 *Question.* If $1 \leq s < \infty$, then for which crescents G is $P^s(\omega) = H^s(G)$, where ω is harmonic measure for G at some z_0 in G?

The following proposition (whose proof is left as an excercise) encourages one to believe that there is an answer to Question 2.1 which solely involves "geometric" conditions.

2.2 PROPOSITION. *Suppose $1 \leq s < \infty$, G is a crescent, ω is harmonic measure for G at z_0 in G and $P^s(\omega) = H^s(G)$. If E is a crescent such that $E \subseteq G$ and $\{\overline{G}\,\hat{}\backslash\overline{G}\} \subseteq \{\overline{E}\,\hat{}\backslash\overline{E}\}$, and if ν is harmonic measure for E at ξ_0 in E, then $P^s(\nu) = H^s(E)$.*

Rather than assaulting Question 2.1 in its full generality, this paper focuses on a collection of "nice" crescents.

Let $\mathcal{F}$ be the collection of all functions f which have the following properties:

(i) $f\colon \mathbf{R} \to \mathbf{R}$, $f(x) = f(-x)$, $f \geq 0$ and $f(x) = 0$ if and only if $x = 0$.

(ii) f is continuously differentiable, f' is nondecreasing and $f'(0) = 0$.

(iii) There exist constants (depending on f) $\beta > \alpha > 1$ and $\delta > 0$ such that $x^\beta \leq f(x) \leq x^\alpha$ whenever $0 < x < \delta$.

The assumptions that f is an even function and that f' is continuous are made for convenience of proof; they can be eliminated. Furthermore, Proposition 2.2 later reveals that, for our purposes, condition (iii) is scarcely a restriction.

If $t > 0$ and $f \in \mathcal{F}$, then let $A(t) = \{z \in \mathbf{C}\colon \mathrm{Im}(z) > 0 \text{ and } |z| < t\}$ and $B(f,t) = \{z = x + iy\colon y \geq f(x) \text{ and } |z| \leq t\}$. Let $\mathcal{C} = \{A(R)\backslash B(f,r)\colon 0 < r < R$

and $f \in \mathcal{F}\}$. $\mathcal{C}$ is seen to be a collection of crescents, each crescent of which lies in the upper half-plane, is symmetric about the imaginary axis and has $z = 0$ as its multiple boundary point. Moreover, $\mathcal{C}$ is substantial in the sense that a wide variety of rates of "thinning", near the multiple boundary point, are represented by crescents in $\mathcal{C}$.

If $f \in \mathcal{F}$, $0 < \theta, \tau < \pi/2$ and n is any positive integer, then let $S(\tau) = \{te^{i\tau} : t > 0\}$ and

$$\xi_n \equiv \xi_n(\theta, f) = \begin{cases} \{x + if(x) \colon x \in \mathbf{R}\} \cap S(\theta/n) & \text{if nonempty,} \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $\mathcal{F}$, ξ_n is always a well-defined complex number. Let $x_n \equiv x_n(\theta, f) = \operatorname{Re}(\xi_n(\theta, f))$ and $\rho_n \equiv \rho_n(\theta, f) = |\xi_n(\theta, f)|$. If $\xi_n \neq 0$, then $\xi_k \neq 0$ for all $k \geq n$ and

$$\rho_n > \rho_{n+1} > \cdots > \rho_{n+j} \to 0 \quad \text{as } j \to \infty.$$

2.3 LEMMA. *If $f \in \mathcal{F}$ and $\int_0^\varepsilon (f')^{-1} dm = \infty$ for some $\varepsilon > 0$, then $\sum_{n=1}^\infty \rho_n(\theta, f) = \infty$ whenever $0 < \theta < \pi/2$.*

PROOF. Suppose that $\sum_{n=1}^\infty \rho_n(\theta, f) < \infty$ for some θ, $0 < \theta < \pi/2$. Hence $\sum_{n=1}^\infty x_n < \infty$ ($x_n \equiv x_n(\theta, f)$). Now, there exists $N \geq 1$ such that for all $n \geq N$, $x_n > 0$. For $n \geq N$, let $t_n \equiv t_n(\theta, f) = \min\{x \colon f'(x) = \theta/n\}$. Since f' is nondecreasing and $f'(0) = f(0) = 0$, it follows that (for $n \geq N$) $f'(x_n) > \theta/n$, and so $t_n < x_n$. Therefore,

$$\sum_{n=N}^\infty t_n(f'(t_{n+1})^{-1} - f'(t_n)^{-1}) = \frac{1}{\theta} \cdot \sum_{n=N}^\infty t_n((n+1) - n) = \frac{1}{\theta} \cdot \sum_{n=N}^\infty t_n < \infty.$$

Since $(f')^{-1}|_{(0,\infty)}$ is nonincreasing, we can conclude that

$$\int_0^\varepsilon (f')^{-1} dm < \infty. \qquad \text{Q.E.D.}$$

2.4 LEMMA. *Suppose that $f \in \mathcal{F}$ and ω is harmonic measure for $G \equiv A(R) \backslash B(f, r) \in \mathcal{C}$ at z_0 in G. If $1 \leq s < \infty$ and $\sum_{n=1}^\infty \rho_n(\theta, f) = \infty$ whenever $0 < \theta < \pi/2$, then $\sum_{n=1}^\infty \|z^{-2n}\|_s^{-1/2n} = \infty$ ($z^{-n} \in L^s(\omega)$ for $n = 1, 2, 3, \ldots$).*

PROOF. We may assume that s is a positive integer. Choose $\theta = \pi/8s$. Let $z = te^{i\varphi}$ and $h_{n,s}(z) = \operatorname{Re}(z^{-2ns}) = t^{-2ns} \cdot \cos(2ns\varphi)$. Now there exists a positive integer N such that $0 < \rho_n(\theta, f) < r$ whenever $n \geq N$. Hence, for $n \geq N$ and $\rho_n \equiv \rho_n(\theta, f)$ ($\theta = \pi/8s$),

$$\begin{aligned}
\|z^{-2n}\|_s^s = \int |z^{-2n}|^s d\omega &= \int_{B(0;\rho_n)} |z^{-2n}|^s d\omega + \int_{\overline{G} \backslash B(0;\rho_n)} |z^{-2n}|^s d\omega \\
&\leq 2 \cdot \int_{B(0;\rho_n)} h_{n,s} d\omega + \int_{\overline{G} \backslash B(0;\rho_n)} |z^{-2n}|^s d\omega \\
&\qquad \text{(since } h_{n,s}(z) \geq |z^{-2n}|^s/2 \text{ for } z \text{ in } B(0;\rho_n) \cap \overline{G}) \\
&\leq 2 \cdot \int h_{n,s} d\omega + 3 \cdot \int_{\overline{G} \backslash B(0;\rho_n)} |z^{-2n}|^s d\omega \\
&\leq 2|z_0|^{-2ns} + 3\rho_n^{-2ns} \qquad \left(\text{since } \int h_{n,s} d\omega = h_{n,s}(z_0)\right) \\
&\leq 3C^{-2ns} \cdot \rho_n^{-2ns},
\end{aligned}$$

where $C = (|z_0|^{-1}+1)^{-1}$. Therefore,

$$\|z^{-2n}\|_s^{-1/2n} \geq C3^{-1/2ns} \cdot \rho_n \geq (C/2) \cdot \rho_n.$$

Thus,

$$\sum_{n=1}^{\infty} \|z^{-2n}\|_s^{-1/2n} \geq \sum_{n=N}^{\infty} \|z^{-2n}\|_s^{-1/2n} \geq \frac{C}{2} \cdot \sum_{n=N}^{\infty} \rho_n = \infty. \quad \square$$

The following theorem is a relative of [**3**, Theorem 2], where the context is polynomial approximation in the mean for area measure on crescents.

2.5 THEOREM. *Suppose that* $f \in \mathcal{F}$, ω *is harmonic measure for* $G \equiv A(R)\backslash B(f,r) \in \mathcal{C}$ *at* z_0 *in* G *and* $1 \leq s < \infty$. *If* $\int_0^{\varepsilon} (f')^{-1} dm = \infty$ *for some* $\varepsilon > 0$, *then* $P^s(\omega) = H^s(G)$.

PROOF. To prove the theorem it is sufficient to show that if $g \in L^s(\omega)^* = L^t(\omega)$ $(1/s+1/t=1)$ and $g \perp P^s(\omega)$, then $g \perp H^s(G)$ $(\cong R^s(\overline{G},\omega))$. That is, if $g \in L^t(\omega)$, $\hat{g}(\xi) \equiv \int g(z)/(z-\xi)\, d\omega(z)$ and $\hat{g}|_{\mathbf{C}\backslash\{\overline{G}\,\hat{}\,\}} \equiv 0$, then $\hat{g}|_{\{\overline{G}\,\hat{}\,\}\backslash\overline{G}} \equiv 0$. In light of this, choose g in $L^t(\omega)$ such that $\hat{g}|_{\mathbf{C}\backslash\{\overline{G}\,\hat{}\,\}} \equiv 0$. Since $f \in \mathcal{F}$, there exists $\alpha > 1$ and δ, $0 < \delta < r/(1+r)$, such that $f(x) \leq x^{\alpha}$ whenever $0 < x < \delta$ (we may assume that $1 < \alpha < 2$). Let $\Omega = \{z = x+iy\colon y > |x|^{(\alpha+1)/2} \text{ and } |z| < \delta\}$.

There exists $k > 0$ such that for all z in $\overline{G}$ and ξ in $\overline{\Omega}$,

$$|z-\xi| \geq k|z|^2. \tag{2.5.1}$$

Moreover, for z in $\overline{G}\backslash\{0\}$, ξ in $\overline{\Omega}$ and any positive integer n,

$$\frac{1}{z-\xi} = \frac{1}{z} + \cdots + \frac{\xi^{n-1}}{z^n} + \frac{\xi^n}{z^n(z-\xi)}.$$

By Lemma 2.4, Runge's theorem, the geometry of G and Lebesgue's dominated convergence theorem, $a_i(z) \equiv \xi^{i-1}/z^i \in P^s(\omega)$ for positive integers i. Since $g \perp P^s(\omega)$, it follows that

$$\hat{g}(\xi) = \xi^n \cdot \int \frac{g(z)}{z^n(z-\xi)}\, d\omega(z).$$

Therefore,

$$\begin{aligned} |\hat{g}(\xi)| &\leq |\xi|^n \cdot \int \frac{|g(z)|}{|z^n||z-\xi|} d\omega(z) \\ &\leq \frac{|\xi|^n}{k} \cdot \int |g(z)||z^{-n-2}| d\omega(z) \qquad \text{(by (2.5.1))} \\ &\leq \frac{|\xi|^n}{k} \cdot \|z^{-n-2}\|_s \cdot \|g\|_t. \end{aligned}$$

Let φ: $\mathbf{D} \to \Omega$ ($\mathbf{D} = \{z\colon |z| < 1\}$) be the Riemann map such that $\varphi(1) = 0$. Let $\psi(w) = (\hat{g} \circ \varphi)(w)$ and note that

$$|\psi(w)| \leq \left| |1-w) \cdot \frac{\varphi(w)}{1-w} \right|^n \cdot \frac{\|g\|_t}{k} \cdot \|z^{-n-2}\|_s.$$

By [**7**, Theorem ix. 9. (ii)],

$$0 < \sup_{w \in \mathbf{D}} \left| \frac{\varphi(w)}{1-w} \right| = M < \infty.$$

Therefore,

$$|\psi(w)| \le |1-w|^n \cdot M^n \cdot \|z^{-n-2}\|_s \cdot \|g\|_t/k.$$

We may assume that $\|g\|_t \le k/\|z^{-2}\|_s$ and that $|\psi(w)| \le 1$ for all w in $\mathbf{D}$. Hence,

$$|\psi(w)| \le |1-w|^n \cdot M^n \cdot \|z^{-n-2}\|_s/\|z^{-2}\|_s,$$

for $n = 0, 1, 2, \ldots$. Let $A_n = M^n \cdot \|z^{-n-2}\|_s/\|z^{-2}\|_s$. An easy exercise (left to the reader) shows that $\{A_n\}_{n=0}^{\infty}$ is a log-convex sequence. In other words $A_0 = 1$ and, for $n \ge 1$, $A_n^2 \le A_{n-1} \cdot A_{n+1}$. Hence, for $n \ge 1$,

$$A_n^{1/n} \le A_{n+1}^{1/(n+1)}. \tag{2.5.2}$$

Now

$$A_n^{-1/n} \ge \frac{1}{M} \cdot \left[\frac{\|z^{-2}\|_s \cdot \|z^{-4}\|_s}{\|z^{-2n}\|_s}\right]^{1/2n}$$

and so, from the hypothesis, Lemmas 2.3 and 2.4, it follows that $\sum_{n=1}^{\infty} A_n^{-1/n} = \infty$. With this fact and (2.5.2), [**2**, p. 50] tells us that $\psi \equiv 0$. Therefore, $\hat{g}|_\Omega \equiv 0$. Yet $\hat{g}$ is analytic off the support of ω. So $\hat{g}|_{\{\overline{G}^\frown\}\setminus\overline{G}} \equiv 0$ and the result follows. □

2.6 THEOREM. *Suppose that* $f \in \mathcal{F}$, ω *is harmonic measure for* $G \equiv A(R)\setminus B(f,r) \in \mathcal{C}$ *at* z_0 *in* G *and* $1 \le s < \infty$. *If* $P^s(\omega) = H^s(G)$, *then*

$$\int_0^{\varepsilon} (f'(x))^{-1} \cdot |\log(x)| dm(x) = \infty \qquad \textit{for } \varepsilon > 0.$$

PROOF. Assume that, for some positive ε,

$$\int_0^{\varepsilon} (f'(x))^{-1} \cdot |\log(x)| dm(x) < \infty.$$

We want to show that $P^s(\omega) \ne H^s(G)$. One may assume that $r = 3$ (exercise). Let ν be harmonic measure for $\operatorname{int}(\overline{G}^\frown)$ at z_0. By [**5**] Szegö's Theorem 8.2, the symmetry of G and Proposition 1.3, if $\omega_\infty \equiv \omega|_{\partial_\infty G}$ and there exists ε, $0 < \varepsilon < 3$, such that $\int_0^\varepsilon \log(d\omega_\infty/d\nu)d\nu > -\infty$, then $P^s(\omega_\infty) \ne R^s(\partial_\infty G, \omega_\infty)$ and hence $P^s(\omega) \ne H^s(G)$. By the nature of ν, this reduces to showing that $\int_0^\varepsilon \log(d\omega_\infty/dm)dm > -\infty$.

With our strategy established, let us turn our attention to a crescent which simplifies estimates; estimates that can be applied to G. Let $E = G \cup \{z\colon \operatorname{Im}(z) \le 0$ and either $|z-1| < \frac{1}{4}$ or $|z+1| < \frac{1}{4}\}$, and let σ be harmonic measure for E at z_0. If $0 < x < \frac{1}{2}$, then let $\theta_x = \tan^{-1}(f'(x))$ and $l_x = \{x + if(x) + re^{i\theta_x} : r \in \mathbf{R}\}$. Let $r_x = l_x \cap \mathbf{R}$, $d_x = r_x + e^{i\theta_x/2} \in G$ and σ_x be harmonic measure for E at d_x (see Figure 1).

If $U_x = \{z \in E\colon 0 < \arg(z - r_x) < \theta_x$ and $|z - r_x| < 2\}$, then let η_x be harmonic measure for U_x at d_x. Notice that $U_x = \{z\colon 0 < \arg(z) < \theta_x$ and $|z| < 2\} + r_x$. (For $\psi_x(z) = (z - r_x)^{\pi/\theta_x}$, let μ_x be harmonic measure for $\psi_x(U_x)$ at $\psi_x(d_x) = i$. Since $[-1,1] \subset \partial\{\psi_x(U_x)\}$ and $\{z\colon \operatorname{Im}(z) > 0$ and $|z| < 2\} \subseteq \psi_x(U_x)$ whenever $0 < x < \frac{1}{2}$, there exists a constant $k > 0$ such that,

$$d\mu_x/dm|_{[0,1/2]} > k, \qquad \text{for } 0 < x < 1/2.) \tag{2.6.1}$$

Now $d_x \to 1 \in E$ as x decreases to zero and so [**4**] Proposition 7.13 provides us with a constant $A > 0$ such that

$$d\sigma/d\sigma_x \ge A, \tag{2.6.2}$$

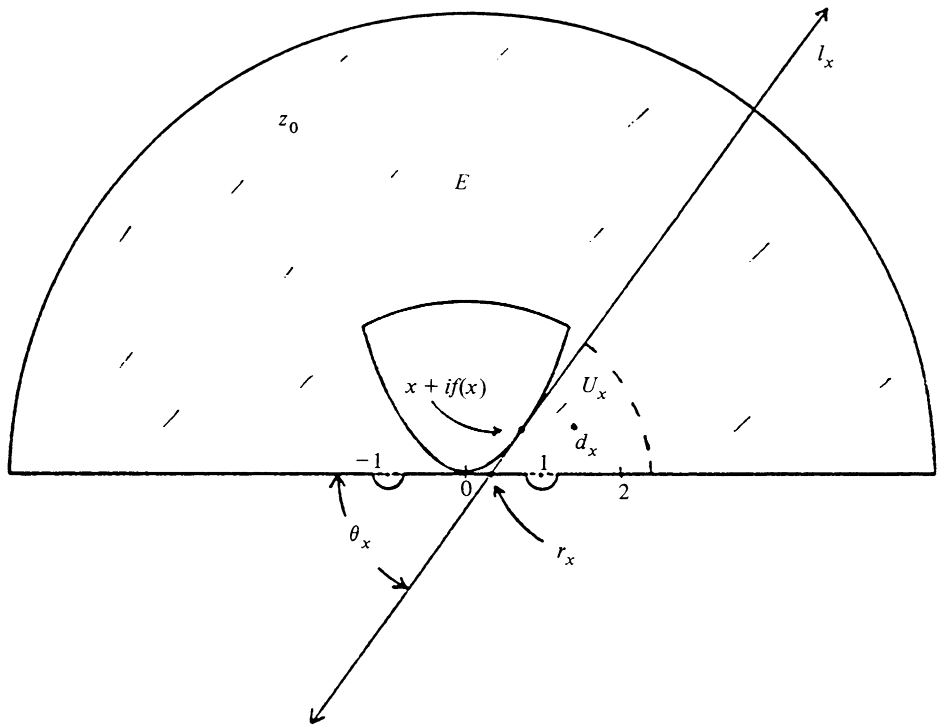

FIGURE 1

whenever $0 < x < \frac{1}{2}$. Since $U_x \subseteq E$,

$$d\sigma_x(z)/d\eta_x \geq 1 \tag{2.6.3}$$

for $0 < x < \frac{1}{2}$ and $r_x \leq z < \frac{1}{2}$. Furthermore, by standard estimates, if $0 < x < \frac{1}{2}$ and $r_x < z < \frac{1}{2}$, then

$$\frac{d\eta_x}{dm}(z) \geq \frac{k\pi}{\theta_x}(z - r_x)^{(\pi/\theta_x)-1} \geq 2k(z - r_x)^{\pi/\theta_x}, \tag{2.6.4}$$

where k is as in (2.6.1).

Note that if $0 < x < y < \frac{1}{2}$, then $0 < r_x \leq r_y < \frac{1}{2}$. Moreover, $r_x \to 0$ as x decreases to zero. If $0 < x < \frac{1}{2}$, then $x \in (r_x, \frac{1}{2})$, $\theta_x \leq \theta_{1/2} < \pi/2$ and $f'(x)/\theta_x = (\tan\ \theta_x)/\theta_x \to 1$ as x decreases to zero. So, there exists a constant $a > 0$ such that if $0 < x < \frac{1}{2}$, then $\theta_x \geq a \cdot f'(x)$ and thus, since $r_x < x < \frac{1}{2}$,

$$d\eta_x(x)/dm > 2k(x - r_x)^{\pi/\theta_x} \geq 2k(x - r_x)^{(\pi/a)\cdot(f'(x))^{-1}}.$$

Since $f(x)/(x - r_x) \to 0$ as x decreases to zero, there exists (by condition (iii) of $\mathcal{F}$) a constant $b > 0$ such that $x - r_x \geq x^b$ for $0 < x < \frac{1}{2}$. Therefore,

$$d\eta_x(x)/dm \geq 2k(x)^{b\pi(f'(x))^{-1}/a} \tag{2.6.5}$$

whenever $0 < x < \frac{1}{2}$.

Combining (2.6.2)–(2.6.5), there exists a constant $B > 0$ such that if $0 < x < \frac{1}{2}$, then

$$d\sigma(x)/dm \geq B(x)^{b\pi(f'(x))^{-1}/a}.$$

By Proposition 1.2, there exists a constant $C > 0$ such that

$$d\omega_\infty(x)/dm = d\omega(x)/dm \geq C(x)^{b\pi(f'(x))^{-1}/a}$$

whenever $0 < x < \frac{1}{2}$. Our initial assumption now tells us that

$$\int_0^\varepsilon \log(d\omega_\infty/dm)dm > -\infty,$$

and so the conclusion follows. □

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